ABSENCE OF EIGENVALUES OF DISSIPATIVE OPERATOR FOR STRICTLY CONVEX OBSTACLES

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Abstract. We study the wave equation in the exterior of a strictly convex bounded domain $K \subset \mathbb{R}^d, d \geq 3$, odd, with dissipative boundary condition $\partial_\nu u - \gamma(x)\partial_t u = 0$ on the boundary Γ and $0 < \gamma(x) < 1$, $\forall x \in \Gamma$. The solutions are described by a contraction semigroup $V(t) = e^{tG}$, $t \geq 0$. In [10] we established that for $\gamma \equiv const$ and $K = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ the operator G has no eigenvalues and we conjectured that the same result holds for every strictly convex obstacle. In this paper we prove this conjecture.

1. Introduction

Let $K \subset \mathbb{R}^d$, $d \geq 3$, d odd, be a bounded non-empty domain with C^∞ strictly convex boundary Γ. Let $\Omega = \mathbb{R}^d \setminus \bar{K}$ be connected and $K \subset \{x \in \mathbb{R}^d : |x| < \rho\}$. Consider the boundary problem

$$\begin{cases} u_{tt} - \Delta_x u = 0 \text{ in } \mathbb{R}_t^+ \times \Omega, \\ \partial_\nu u - \gamma(x)\partial_t u = 0 \text{ on } \mathbb{R}_t^+ \times \Gamma, \\ u(0,x) = f_1,\ u_t(0,x) = f_2 \end{cases} \tag{1.1}$$

with initial data $(f_1, f_2) \in \mathcal{H} = H^1(\Omega) \times L^2(\Omega)$. Here $\nu(x)$ is the unit outward normal at $x \in \Gamma$ pointing into Ω and $\gamma(x) > 0$ is a C^∞ function on Γ. Introduce the contraction semi-group $V(t)$ in $\mathcal{H}$ with generator

$$G = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}.$$

The operator G has domain $\mathcal{D}$ given by the closure in the graph norm

$$|||f||| = (\|f\|_{\mathcal{H}}^2 + \|Gf\|_{\mathcal{H}}^2)^{1/2}$$

of functions $f = (f_1, f_2) \in C_{(0)}^\infty(\mathbb{R}^d) \times C_{(0)}^\infty(\mathbb{R}^d)$ satisfying the boundary condition $\partial_\nu f_1 - \gamma f_2 = 0$ on Γ.

The solution of the problem (1.1) has the form $V(t)f = e^{tG}f$, $t \geq 0$. The point spectrum $\sigma_p(G)$ of G in $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ is formed by isolated eigenvalues with finite multiplicity, $\sigma_p(G) \cap \mathbf{i}\mathbb{R} = \emptyset$ and G has continuous spectrum $\sigma_c(G) = \mathbf{i}\mathbb{R}$ (see Section 8, [6]). Notice that if $Gf = \lambda f$ with $0 \neq f \in \mathcal{D}$, $\operatorname{Re}\lambda < 0$, we have solution $u(t,x) = V(t)f = e^{\lambda t}f(x)$ of (1.1) with exponentially decreasing global energy. Such solutions are called asymptotically disappearing and they are important for scattering problems (see for more details [8], [10]). The location in $\mathbb{C}_-$ of the eigenvalues of G has been studied in [8] improving previous results of Majda [7]. In the special case when $\gamma \equiv 1$, and K is the unit ball $B_3 = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ it was proved in [8] that G has no eigenvalues. The case $\gamma(x) > 1$, $\forall x \in \Gamma$ has been examined in [8] and a Weyl formula for the counting function of the eigenvalues of G close to the negative real axis has been established in [9].

Throughout this paper we assume that $0 < \gamma(x) < 1$, $\forall x \in \Gamma$. This case has been studied in [8], [10]. In particular, for $\gamma \equiv const$ and $K = B_3$ in Appendix, [10] it was proved that G has no eigenvalues. We conjectured in [10] that the same is true for every strictly convex obstacle. Our main result is the following

Theorem 1.1. *For strictly convex obstacles the operator G has no eigenvalues λ with* $\operatorname{Re}\lambda < 0$.

The eigenvalues of G in $\mathbb{C}_-$ are incoming functions (see Section 2 for the definition of outgoing/incoming functions). The absence of eigenvalues of G for $\operatorname{Re}\lambda > 0$ is based on the representation of the outgoing resolvent

$$(G-\lambda)^{-1} = -\int_0^\infty e^{-\lambda t} e^{tG} dt, \ \operatorname{Re}\lambda > 0 \tag{1.2}$$

which is analytic for $\operatorname{Re}\lambda > 0$. For the incoming eigenvalues it should be useful to have a similar representation with integral on $(-\infty, 0]$ involving some evolution operator. However, the problem

$$\begin{cases}(\partial_t^2 - \Delta_x)f = 0 \text{ in } \mathbb{R}_t^- \times \Omega,\\ \partial_\nu f - \gamma(x)\partial_t f = 0 \text{ on } \mathbb{R}_t^- \times \Gamma,\\ f(0,x) = 0, \ f_t(0,x) = \chi(x)\end{cases} \tag{1.3}$$

is not L^2 well posed (see Theorem 1 in [7]). On the other hand, the problem (1.3) is well posed with some loss of regularity (see [4], [5]) and for $\chi \in H^m_{comp}(\Omega)$, $m > d/2+3$ it is possible to obtain a solution of (1.3) with the property $e^{c_{m-3/2}t} f(t,x) \in H^{m-2}(\mathbb{R}_t^- \times \Omega)$ with some large $c_{m-3/2} > 0$ (see Proposition 3.2). The idea is to consider the incoming function

$$w(x,\lambda) = \int_{-\infty}^0 e^{-\lambda t} P(t)\chi dt, \ \operatorname{Re}\lambda \le -c_{m-3/2},$$

where $P(t)\chi$ is a solution of (1.3) and to modify it to obtain an approximative resolvent $R_a^\sharp(\lambda)$ of $(G-\lambda)^{-1}$.

We expose below the strategy of the proof of Theorem 1.1. In Section 3 we examine the operator $\mathcal{C}(\lambda) = N(\lambda) - \lambda\gamma(x)$, where $N(\lambda) : H^s(\Gamma) \to H^{s-1}(\Gamma)$ is the Dirichlet-to-Neumann operator defined in Section 2. We study the existence of $\mathcal{C}(\lambda)^{-1}$ for $\operatorname{Re}\lambda \le -c_0 < 0$. For this purpose we apply a weak version of Theorem 2.1 in [4] concerning only $N(\lambda)$. We expect that it is possible to obtain the existence of $\mathcal{C}(\lambda)^{-1}$ for large negative $\operatorname{Re}\lambda$ for general obstacles and we will study this problem in a further work. In Proposition 3.2 we justify the existence of an operator $P(t)$ satisfying (3.13). In Section 4 we follow the approach of the proof of Theorem 4.43 in [2] by using a cut-off function $\zeta_a(t,x)$. Let $\chi_a \in H^m_{comp}(\Omega)$, $\chi_a(x) \equiv 1$ for $|x| \le \rho + 2\varepsilon$, $\varepsilon > 0$. We study the operator $\zeta_a P(t)\chi_a$ which satisfies

$$\begin{cases}(\partial_t^2 - \Delta_x)(\zeta_a P(t)\chi_a) =: F_a(t),\\ (\partial_\nu - \gamma(x)\partial_t)(\zeta_a P(t)\chi_a)|_{\mathbb{R}_t^- \times \Gamma} =: J_a(t).\end{cases}$$

To treat the remainder operators $F_a(t)$ and $J_a(t)$, we define an approximative resolvent

$$R_a^\sharp(\lambda) = \int_{-\infty}^0 e^{-\lambda t}\Big[\zeta_a P(t)\chi_a + (1-\chi_c)W_a(t) - B_a(t)\Big]dt, \ \operatorname{Re}\lambda \le 0.$$

with cut-off function $\chi_c(x)$ and suitable correction operators $W_a(t)$, $B_a(t)$. The kernels of these operators have compact support with respect to t and this makes possible to extend analytically $R_a^\sharp(\lambda)$ from $\operatorname{Re}\lambda \leq -c_{m-3/2}$ to $\operatorname{Re}\lambda < 0$. The operator $R_a^\sharp(\lambda)$ satisfies $(\Delta - \lambda^2)R_a^\sharp(\lambda) = \chi_a(\mathrm{Id} + L(\lambda))$ and for sufficiently large $|\lambda| \geq A_0$ we obtain the existence of $(\mathrm{Id} + L(\lambda))^{-1}$. This leads to the absence of eigenvalues λ of G with $|\lambda| \geq A_0$. Finally, Section 5 is devoted to the absence of eigenvalues with $|\lambda| \leq A_0$. For $\varepsilon \in [0,1]$ we introduce the operator $\mathcal{C}_\varepsilon(\lambda) = N(\lambda) - \lambda\varepsilon\gamma(x)$ and the generator G_ε related to problem (1.1) with boundary condition $(\partial_\nu u - \varepsilon\gamma\partial_t u)|_\Gamma = 0$. We introduce the bounded set $\omega = \{z \in \mathbb{C} : -c_0 \leq \operatorname{Re}\lambda \leq 0, |\operatorname{Im} z| \leq A_0\}$ and define $\eta = \sup\{\varepsilon : \varepsilon \in [0,1], G_\varepsilon \text{ has no eigenvalues in } \omega\}$. The problem is reduced to the analysis of the case when G_η has an eigenvalue $\lambda_0 \in \omega$. By a perturbation argument for G_ε and $\eta - \varepsilon > 0$ sufficiently small we show that this is impossible.

2. Preliminaries

In this section we collect some facts from [8], [10]. For $\lambda \in \mathbb{C}$ introduce the exterior Dirichlet-to-Neumann map

$$N(\lambda) : H^s(\Gamma) \ni f \longrightarrow \partial_\nu u|_\Gamma \in H^{s-1}(\Gamma),$$

where $u = K(\lambda)f$ is the solution of the problem

$$\begin{cases} (-\Delta + \lambda^2)K(\lambda)f = 0 \text{ in } \Omega, \\ K(\lambda)f = f \text{ on } \Gamma, \\ K(\lambda)f : \lambda - \text{incoming}. \end{cases} \tag{2.1}$$

A function $u(x)$ is called λ-*outgoing* (λ-*incoming*) if there exists $R > \rho$ and $g \in L^2_{comp}(\mathbb{R}^d)$ such that

$$u(x) = R_0^\pm(\lambda)g, \ |x| \geq R,$$

where $R_0^\pm(\lambda) = (-\Delta_0 + \lambda^2)^{-1}$ is the outgoing (+) (incoming (-)) resolvent of the free Laplacian $-\Delta_0$ in $\mathbb{R}^d$. The resolvents $R_0^\pm(\lambda)$ are analytic in $\mathbb{C}$ and they have kernels

$$R_0^\pm(\lambda, x, y) = \frac{(-1)^{(d-1)/2}}{2(2\pi)^{(d-1)/2}}\Big(\frac{1}{r}\partial_r\Big)^{(d-3)/2}\Big(\frac{e^{\mp\lambda r}}{r}\Big)\Big|_{r=|x-y|}.$$

If for $\operatorname{Re}\lambda < 0$ we have $G(f_1, f_2) = \lambda(f_1, f_2)$, then $f_1 \in H^2(\Omega)$ is λ-incoming solution of $(-\Delta + \lambda^2)f_1 = 0$ (see [6] and Section 1 in [10]). Let $R_D(\lambda) = (-\Delta_D + \lambda^2)^{-1}$ be the incoming resolvent of the Dirichlet Laplacian Δ_D in Ω with domain $\mathbf{D} = H^2(\Omega) \cap H_0^1(\Omega)$ which is analytic for $\lambda \in \mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}$. Let

$$\mathbf{D}_{loc} = \{u \in L^2_{loc}(\Omega) : \chi(x) \in C_0^\infty(\mathbb{R}^d), \chi(x) \equiv 1 in\, a\, neighborhood\, of\, \bar{K} \Rightarrow \chi u \in \mathbf{D}\}.$$

The incoming resolvent has meromorphic extension

$$R_D(\lambda) : \ L^2_{comp}(\Omega) \to \mathbf{D}_{loc}$$

from $\mathbb{C}_-$ to $\mathbb{C}$. The solution of the problem (2.1) with $f \in H^{3/2}(\Gamma)$ has the form

$$u = e(f) + R_D(\lambda)((\Delta - \lambda^2)(e(f)), \tag{2.2}$$

where $e(f) : H^{3/2}(\Gamma) \ni f \to e(f) \in H^2_{comp}(\Omega)$ is an extension operator. Clearly, we may find $\partial_\nu u|_\Gamma$ by applying (2.2). By (2.2) we conclude that $N(\lambda)$ is analytic in $\bar{\mathbb{C}}_-$.

We write the boundary condition in (1.1) as

$$\mathcal{C}(\lambda)v := (N(\lambda) - \lambda\gamma)v = \Big(\mathrm{Id} - \lambda\gamma N(\lambda)^{-1}\Big)N(\lambda)v = 0,\; v = f_1|_\Gamma \in H^{3/2}(\Gamma).$$

For $\operatorname{Re}\lambda \leq 0$ the operator $\mathcal{C}(\lambda):\; H^s(\Gamma) \to H^{s-1}(\Gamma)$ has the same singularities as $N(\lambda)$. On the other hand, $N(\lambda)^{-1}:\; H^s(\Gamma) \to H^{s+1}(\Gamma)$ is compact operator and by analytic Fredholm theorem (see [8], [10]), the operator $\mathcal{C}(\lambda)$ is a meromorphic operator valued function for $\operatorname{Re}\lambda \leq 0$. Notice that if $\mathcal{C}(\lambda)$ is analytic at λ_0, then $\mathcal{C}(\lambda_0) : H^s(\Gamma) \to H^{s+1}(\Gamma)$. Here and below a meromorphic operator valued function $B(z)$ means that $B(z)$ have Laurent expansion with finite number negative powers of z and coefficients having finite rank. In [9] it was proved that we can extend the incoming resolvent $(G-\lambda)^{-1},\ \lambda \notin \sigma_p(G)$, $\operatorname{Re}\lambda < 0$ as meromorphic function $(G-\lambda)^{-1}:\; \mathcal{H}_{comp} \to \mathcal{D}_{loc}$ for $\lambda \in \mathbb{C}$, where

$$\mathcal{D}_{loc} = \{u \in \mathcal{H}_{loc},\; \chi(x) \in C_0^\infty(\mathbb{R}^d),\; \chi(x) \equiv 1\, in\, a\, neighborhood\, of\, \bar{K} \;\Rightarrow \chi u \in \mathcal{D}\}.$$

Let $\begin{pmatrix} u \\ w \end{pmatrix} = (G-\lambda)^{-1}\begin{pmatrix} f \\ g \end{pmatrix}$ with $(f,g) \in \mathcal{H}_{comp}$. It was established in [8], [10] that $(G-\lambda)^{-1}$ is analytic at λ_0 if and only if $\mathcal{C}(\lambda_0)$ is invertible and $\mathcal{C}(\lambda_0)^{-1} : H^s(\Gamma) \to H^{s+1}(\Gamma)$ is analytic at λ_0. Moreover, if $\mathcal{C}(\lambda)^{-1}$ is analytic, $w = \lambda u + f$, where

$$u(x,\lambda) = -R_D(\lambda)(g+\lambda f) + K(\lambda)\mathcal{C}(\lambda)^{-1}\Big[\partial_\nu\Big(R_D(\lambda)(g+\lambda f)\Big)\Big|_\Gamma + \gamma f|_\Gamma\Big] \tag{2.3}$$

is a solution of the problem

$$\begin{cases} (\Delta - \lambda^2)u = g + \lambda f \text{ in } \Omega, \\ (\partial_\nu - \gamma(x)\lambda)u|_\Gamma = \gamma f|_\Gamma, \\ u : \lambda - \text{incoming}. \end{cases} \tag{2.4}$$

Notice that for $(f,g) \in \mathcal{H}_{comp}$ we obtain $u(x,\lambda) \in H^2(\Omega)$. Indeed,

$$\mathcal{C}(\lambda)^{-1}\Big[\partial_\nu R_D(\lambda)(g+\lambda f)\Big|_\Gamma + \gamma f|_\Gamma\Big] \in H^{3/2}(\Gamma)$$

and applying $K(\lambda)$, we obtain the result.

It is easy to see that if $\mathcal{C}(\lambda)^{-1}:\; H^s(\Gamma) \to H^{s+1}(\Gamma)$ for some $s_0 \geq 0$ has a pole $\lambda \in \bar{\mathbb{C}}_-$, then this operator has a pole λ_0 for every $s \geq 0$. This follows from the fact that there exists $0 \neq v \in H^s(\Gamma)$ such that $\mathcal{C}(\lambda_0)v = (\mathrm{Id} - \lambda_0\gamma N(\lambda_0)^{-1})v = 0$. Since $N(\lambda_0)^{-1} : H^s(\Gamma) \to H^{s+1}(\Gamma)$ is bounded, this implies $v \in C^\infty(\Gamma)$ and $(\mathrm{Id} - \lambda_0\gamma N(\lambda_0)^{-1})$ is not invertible in $H^s(\Gamma)$ for all $s \geq 0$. In Section 4 we need the following

Lemma 2.1. *Assume that there exist $a_1 > \rho$ and $k \geq 2$ such that for $f = 0$ and every $g \in H^k(\Omega)$ with* supp $g \subset B(0,a_1)$ *there exists a solution $u(x,\lambda)$ of (2.4) which for $x \in \Omega \cap \{x \in \mathbb{R}^d :\; |x| \leq a,\; a > \rho\}$ is analytic in a neighbourhood $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \delta\}$ of $\lambda_0 \in \mathbb{C}_-$. Then the operator $\mathcal{C}(\lambda)^{-1} : H^{k+1}(\Gamma) \to H^{k+2}(\Gamma)$ is analytic at λ_0.*

Proof. Suppose that $\mathcal{C}(\lambda)^{-1}$ has a pole at λ_0. Then there exist $h \in H^{k+1}(\Gamma)$, $0 \neq h_0 \in H^{k+2}(\Gamma)$ and $m \geq 1$ such that for $0 \neq |\lambda - \lambda_0| < \varepsilon$ with sufficiently small $0 < \varepsilon \leq \delta$ we have

$$\mathcal{C}(\lambda)^{-1}h = \frac{h_0}{(\lambda-\lambda_0)^m} + \text{lower order singularities}.$$

Let $q = N(\lambda_0)^{-1}h \in H^{k+2}(\Gamma)$. It is easy to construct $\tilde{g} \in H^{k+2}(\Omega)$ with supp $\tilde{g} \subset B(0, a_1)$ such that $\tilde{g}|_\Gamma = q$, $\partial_\nu \tilde{g}|_\Gamma = 0$. To do this, by using a partition of unity on Γ, we pass to a construction in a neighbourhood $\mathcal{U} \subset \mathbb{R}^d$ of a point $x_0 \in \Gamma$. Consider in $\mathcal{U}$ local normal coordinates $(y_1, ..., y_d) = (y_1, y')$ such that

$$x = \alpha(y_1, y') = \beta(y') + y_1\nu(y'),$$

where $y_1 = \text{dist}\,(x, \Gamma)$ and $\nu(y')$ is an extension of unit normal vector to unit vector field. Then the boundary Γ is given by $y_1 = 0$ and $\partial_\nu v = \partial_{y_1} v(\alpha(y_1, y))$. If $q(\alpha(0, y')) = s(y')$, we take $\tilde{g}(y) = s(y')$. Next for $f = 0$ and $g = (\Delta - \lambda_0^2)\tilde{g} \in H^k(\Omega)$ by our assumption there exists a solution $u(x, \lambda)$ of (2.4) which is analytic for $x \in \Omega,\ |x| \leq a,\ |\lambda - \lambda_0| < \delta$. The function $v = R_D(\lambda_0)g$ satisfies

$$(-\Delta + \lambda_0^2)(v + \tilde{g}) = 0,\ (v + \tilde{g})|_\Gamma = q,$$

hence $\partial_\nu v|_\Gamma = \partial_\nu(v + \tilde{g})|_\Gamma = N(\lambda_0)q = h$. Then in the representation (2.3) for $u(x, \lambda)|_\Gamma$ we will have a singular term given by $\mathcal{C}(\lambda)^{-1}\partial_\nu R_D(\lambda_0)g)|_\Gamma$ and we obtain a contradiction with the analyticity of $u(x, \lambda)$ for $|x| \leq a$. $\square$

Remark 2.1. *By the same argument we may construct a modification $\tilde{g} \in H^m(\Omega)$ of $g \in H^m(\Omega)$ such that $\tilde{g}|_\Gamma = 0,\ \partial_\nu \tilde{g}|_\Gamma = g|_\Gamma$. To do this, in local normal coordinates (y_1, y') we take $\tilde{g}(y) = y_1 g(\alpha(y_1, y'))$.*

3. Existence and estimate of $\mathcal{C}(\lambda)^{-1}$

3.1. **Analyticity of $\mathcal{C}(\lambda)^{-1}$.** Throughout the paper we will use the notation $\lambda = \mu + \mathbf{i}k$, μ, $k \in \mathbb{R}$. In this subsection we study the analyticity of $\mathcal{C}(\lambda)^{-1}$ for sufficiently negative μ. Denote by $(.,.)_s$, $\|.\|_s$ are the scalar product and the norm in $H^s(\Gamma)$, respectively. Consider the problem

$$\begin{cases} (-\Delta + \lambda^2)K_1(\lambda)f = 0 \text{ in } \Omega, \\ K_1(\lambda)f = f \text{ on } \Gamma, \\ K_1(\lambda)f : \lambda - \text{outgoing}. \end{cases}$$

and the Dirichlet-to-Neumann operator $N_1(\lambda)f = \partial_\nu K_1 f(\lambda)|_\Gamma$. Ikawa established the following

Theorem 3.1 (Theorem 2.1 in [4])**.** *Let K be a strictly convex obstacle in $\mathbb{R}^3$. Then there exist constants $r > 0, C_m > 0$ such that for every $m \in \mathbb{N}$ and $\mu \geq r$ we have*

$$-\operatorname{Re}(N_1(\lambda)\varphi, \varphi)_m \geq (\mu - C_m)\|\varphi\|_m^2,\ \ \forall \varphi \in C^\infty(\Gamma) \tag{3.1}$$

with C_m independent of λ and φ.

After the change $\eta = -\lambda$, we deduce for operator $N(\lambda)$ related to (2.1) and $\mu \leq -r$ the estimate

$$-\operatorname{Re}(N(\lambda)\varphi, \varphi)_m \geq (|\mu| - C_m)\|\varphi\|_m^2,\ \ \forall \varphi \in C^\infty(\Gamma). \tag{3.2}$$

Remark 3.2. *We apply the estimate (3.1) in [4] with $\gamma \equiv 0$. This estimate can be proved for odd dimensions $d > 3$ by similar argument. Notice that some estimates of the operator $\frac{N(\lambda)}{\lambda}$ in suitable regions have been established in Section 4 in [10].*

Let $\delta = \max_{x\in\Gamma} \gamma(x) < 1$. For $m = 0$ we have $(\gamma\varphi, \varphi)_0 \leq \delta\|\varphi\|_0^2$ and we obtain the estimate

$$-\operatorname{Re}((N(\lambda) - \lambda\gamma(x))\varphi, \varphi)_0 \geq \Big((1 - \delta)|\mu| - C_0\Big)\|\varphi\|_0^2.$$

Setting $\alpha = 1 - \delta > 0$, $c_0 = \max\{\frac{2C_0}{\alpha}, r\}$, for $\mu \leq -c_0$ we may absorb the term $C_0\|\varphi\|_0^2$ and obtain

$$\|\mathcal{C}(\lambda)\varphi\|_0 \geq \frac{\alpha|\mu|}{2}\|\varphi\|_0, \ \forall \varphi \in C^\infty(\Gamma). \tag{3.3}$$

In the following we suppose that $\mu \leq -c_0$. Set $M(\lambda) = \Big(Id - \lambda\gamma N^{-1}(\lambda)\Big)$ and note that $\mathcal{C}(\lambda) = M(\lambda)N(\lambda)$. If for $s \geq 0$ the operator $\mathcal{C}(\lambda)^{-1} : H^s(\Gamma) \to H^{s+1}(\Gamma)$ has a pole at λ_0, then $M(\lambda_0) : H^s(\Gamma) \to H^s(\Gamma)$ is not invertible and there exists $0 \neq \varphi_0 \in L^2(\Gamma)$ such that $\mathcal{C}(\lambda_0)\varphi_0 = M(\lambda_0)\varphi_0 = 0$. Since $N(\lambda_0)^{-1} : H^s(\Gamma) \to H^{s+1}(\Gamma)$, we conclude that $\varphi_0 \in C^\infty(\Gamma)$. This leads to a contradiction with (3.3), hence $\mathcal{C}(\lambda)^{-1}$ is analytic for $\mu \leq -c_0$ and c_0 is independent of s.

Remark 3.3. *The invertibility of $\mathcal{C}(\lambda)$ for $s \in \mathbb{N}$ and $\mu \geq -c_0$ has been proved in* [4] *following Lemma* 3.3 *in* [3]. *Our argument above is short and we use* (3.2) *only for $m = 0$.*

Now we will estimate the norm of $\mathcal{C}(\lambda)^{-1} : H^s(\Gamma) \to H^s(\Gamma)$. Let $h \in L^2(\Gamma)$ and let $C(\lambda)g = h$ with $g \in H^1(\Gamma)$. (We omit in the notations the dependence of g of λ). Choose a sequence $\varphi_j \to_{j\to\infty} g$ in $H^1(\Gamma)$ with $\varphi_j \in C^\infty(\Gamma)$. Therefore

$$\|C(\lambda)(\varphi_j - g)\|_0 \leq B_0(\lambda)\|\varphi_j - g\|_1$$

and $C(\lambda)\varphi_j \to C(\lambda)g$ in $L^2(\Gamma)$. On the other hand, $\|\varphi_j\|_0 \to \|g\|_0$. We apply (3.3) with φ_j and passing to limit, we deduce

$$\|\mathcal{C}(\lambda)^{-1}h\|_0 \leq \frac{2}{\alpha|\mu|}\|h\|_0, \ \forall h \in L^2(\Gamma). \tag{3.4}$$

Next assume that for $\mu \leq -c_m \leq -c_0$ we have the estimate

$$\|\mathcal{C}(\lambda)^{-1}h\|_m \leq \frac{B_m}{\alpha|\mu|}\|h\|_m, \ \forall h \in H^m(\Gamma). \tag{3.5}$$

Applying (3.2), we get

$$-\operatorname{Re}((N(\lambda) - \lambda\gamma(x))\varphi, \varphi)_{m+1} \geq (|\mu| - C_{m+1})\|\varphi\|_{m+1}^2 + \mu(\gamma\varphi, \varphi)_{m+1}.$$

On the other hand,

$$|\mu(\gamma\varphi, \varphi)_{m+1}| \leq |\mu|\delta\|\varphi\|_{m+1}^2 + A_m|\mu|\|\varphi\|_m\|\varphi\|_{m+1},$$

where A_m depends of the norms of the derivatives of $\gamma(x)$ of order $j \leq m$. The above estimates imply

$$A_m|\mu|\|\varphi\|_m\|\varphi\|_{m+1} + \|\mathcal{C}(\lambda)\varphi\|_{m+1}\|\varphi\|_{m+1} \geq (\alpha|\mu| - C_{m+1})\|\varphi\|_{m+1}^2.$$

We estimate $|\mu|\|\varphi\|_m$ from (3.5) and obtain

$$(A_m B_m \alpha^{-1} + 1)\|\mathcal{C}(\lambda)\varphi\|_{m+1} \geq (\alpha|\mu| - C_{m+1})\|\varphi\|_{m+1}.$$

Choosing $c_{m+1} = \max\{\frac{2C_{m+1}}{\alpha}, c_m\}$ and repeating the above argument with an approximation $\varphi_j \to g$ in $H^{m+2}(\Gamma)$, we conclude for $\mu \leq -c_{m+1}$ that

$$\|\mathcal{C}(\lambda)^{-1}h\|_{m+1} \leq \frac{B_{m+1}}{\alpha|\mu|}\|h\|_{m+1}, \ \forall h \in H^{m+1}(\Gamma).$$

By iteration we establish this estimate for every $m \in \mathbb{N}$. Next applying an interpolation argument for the spaces $H^m(\Gamma)$ and $H^{m+1}(\Gamma)$, we extend it to $H^s(\Gamma)$, $m < s < m + 1$ and obtain the following

Proposition 3.1. *For every $s \geq 0$ there exist constants $c_s > 0$, $B_s > 0$ such that $\mathcal{C}(\lambda)^{-1} : H^s(\Gamma) \to H^{s+1}(\Gamma)$ is analytic for $\mu \leq -c_s$ and in this domain we have*

$$\|\mathcal{C}(\lambda)^{-1}h\|_s \leq \frac{B_s}{\alpha|\mu|}\|h\|_s,\ \forall h \in H^s(\Gamma). \tag{3.6}$$

3.2. **Existence of solution of the problem (1.3).** Let $\chi \in H^m(\Omega)$ with $m \in \mathbb{N}$, $m > d/2+3$ and $\operatorname{supp}\chi \subset B(0,a) = \{x : |x| \leq a\}$, $a > \rho$. Consider an extension $\tilde{\chi} \in H^m(\mathbb{R}^d)$ such that $\tilde{\chi}|_\Omega = \chi$ and $\|\tilde{\chi}\|_{H^m(\mathbb{R}^d)} \leq M\|\chi\|_{H^m(\Omega)}$. Let $g_0(t,x)$ be the solution of

$$\begin{cases}(\partial_t^2 - \Delta_x)g_0 = 0 \text{ in } \mathbb{R}\times\Omega,\\ g_0(0,x) = 0,\ \partial_t g_0(0,x) = \tilde{\chi}(x).\end{cases} \tag{3.7}$$

The function g_0 has the representation $g_0 = \frac{\sin(t\sqrt{-\Delta_0})}{\sqrt{-\Delta_0}}\tilde{\chi}$, where $-\Delta_0$ is the free Laplacian in Ω. Set

$$w_0(\lambda,x) = \int_{-\infty}^{\infty} e^{-\lambda t}H(t)g_0(t,x)dt,\ \operatorname{Re}\lambda \leq 0,$$

where throughout our paper we use the Heaviside function $H(t) = 1$ for $t \leq 0$ and $H(t) = 0$ for $t > 0$. We obtain $(-\Delta+\lambda^2)w_0 = \tilde{\chi}$ and

$$q(\lambda,x) = (\partial_\nu - \lambda\gamma(x))w_0(\lambda,x)|_\Gamma = \int_{-\infty}^{\infty} e^{-\lambda t}H(t)(\partial_\nu - \gamma\partial_t)g_0(t,x)|_\Gamma dt. \tag{3.8}$$

Thus $w_0(\lambda,x) = R_0(\lambda)\tilde{\chi} \in H^{m+2}(\mathbb{R}^d)$, where $R_0(\lambda) = (-\Delta_0+\lambda^2)^{-1}$ is the incoming resolvent of $-\Delta_0$ and $w_0(\lambda,x)$ is analytic for $\operatorname{Re}\lambda \leq 0$. Let $\|.\|_{H^m(B(0,b))} = \|.\|_{m,b}$, $b > \rho$. It is well known that

$$\|R_0(\lambda)\chi\|_{k,b} \leq C_{a,b}|\lambda|^{k-1}\|\tilde{\chi}\|_{0,a},\ \ \operatorname{Re}\lambda \leq 0, |\lambda| \geq L_0 > 1,\ k = 0,1. \tag{3.9}$$

In fact, for $|x| \leq b$, the sharp Huyghens principle yields $g_0(t,x) = 0$ for $|t| > a+b$ and (3.9) follows after an integration by parts. The estimate (3.9) implies

$$\|\partial_x^\alpha w_0\|_{k,b} \leq C_{a,b}|\lambda|^{k-1}\|\chi\|_{m,a},\ \ |\alpha| \leq m,\ k = 0,1$$

and from the equation $\Delta\partial_x^\alpha w_0 - \lambda^2\partial_x^\alpha w_0 = \partial_x^\alpha\tilde{\chi}$ we deduce

$$\|\partial_x^\alpha w_0\|_{m-1,b} \leq C_{m,a,b}|\lambda|^{k-2}\|\chi\|_{m,a},\ |\alpha| = k = 0,1.$$

Since $m-3 > d/2$, the functions $w_0(\lambda,x)$ and $\partial_{x_j}w_0(\lambda,x), j = 1,...,d$ are continuous in $|x| \leq b$ with uniform bound $\mathcal{O}(|\lambda|^{-2})$ and $\mathcal{O}(|\lambda|^{-1})$, respectively, and

$$\|q(\lambda,x)\|_{H^{m-3/2}(\Gamma)} \leq D_m|\lambda|^{-1}\|\chi\|_{m,a}.$$

Below up to end of this subsection we suppose that $\mu \leq -c_{m-3/2}$. Introduce the analytic function

$$h(\lambda,x) = \mathcal{C}(\lambda)^{-1}q(\lambda,x) \in H^{m-1/2}(\Gamma).$$

Applying the estimates (3.6), we deduce

$$\|h(\lambda,x)\|_{H^{m-3/2}(\Gamma)} \leq B_{m-3/2}D_m(\alpha|\lambda\mu|)^{-1}\|\chi\|_{m,a}.$$

Next, let $w_1(\lambda,x) = K(\lambda)h(\lambda,x) \in H^{m-2}(\Omega)$ be the solution of the Dirichlet problem (2.1) with $f = h(\lambda,x)$. Clearly,

$$(\partial_\nu - \lambda\gamma(x))w_1(\lambda,x)|_\Gamma = (N(\lambda) - \lambda\gamma(x))h(\lambda,x) = q(\lambda,x). \tag{3.10}$$

On the other hand, for $w_1(\lambda, x)$ we have the estimates

$$\sum_{k+j\leq m-2} |\lambda|^j \|w_1\|_{H^k(\Omega)} \leq K_m \|h\|_{H^{m-3/2}(\Gamma)} \leq M_m(\alpha|\lambda\mu|)^{-1}\|\chi\|_{m,a}. \tag{3.11}$$

Consequently, the inclusion $H^{m-3}(\Omega) \subset C(\Omega)$ implies

$$|w_1(\lambda, x)| \leq M_m |\lambda|^{-2}|\mu|^{-1}\|\chi\|_{m,a}$$

uniformly with respect to $\mu \leq -c_{m-3/2}$ and $x \in \Omega$. For $(t, x) \in \mathbb{R}_t \times \bar{\Omega}$ and $c \geq c_{m-3/2}$ define

$$g_1(t, x) = \frac{1}{2\pi i}\int_{\mu=-c} e^{\lambda t} w_1(\lambda, x) d\lambda = \frac{e^{-ct}}{2\pi}\int_{-\infty}^{\infty} e^{\mathbf{i}kt} w_1(-c + \mathbf{i}k, x) dk.$$

We may shift the contour of integration to the left, hence $g_1(\lambda, x)$ does not depend of the choice of $c \geq c_{m-3/2}$. It is easy to see that $g_1(t, x) \equiv 0$ for $t \geq 0$. For fixed $t_0 \geq 0, x_0 \in \Omega$ one obtains

$$|g_1(t_0, x_0)| \leq \frac{e^{-ct_0}}{2\pi}\int_{-\infty}^{\infty} |w_1(-c+\mathbf{i}k, x_0)| dk \leq \frac{Ae^{-ct_0}}{c}\int_{-\infty}^{\infty} \frac{1}{c_{m-3/2}^2 + k^2} dk \leq \frac{Be^{-ct_0}}{c}$$

and taking $c \to \infty$, we conclude that $g_1(t_0, x_0) = 0$.

According to (3.11), we get

$$\|w_1\|_{H^{m-2-j}(\Omega)} = \mathcal{O}_m(|\mu|^{-1}|\lambda|^{-j-1}),\ j = 0, ..., m-2.$$

In particular, since $m \geq 5$, we have $\|\Delta w_1\|_{L^2} \leq \|w_1\|_{H^{m-3}} = \mathcal{O}_m(|\mu|^{-1}|\lambda|^{-2})$. Therefore,

$$(2\pi i)\Delta g_1 = \int_{\mu=-c} e^{\lambda t}\Delta w_1(\lambda, x) d\lambda = \lim_{R\to\infty}\int_{-c-iR}^{-c+iR} e^{\lambda t}\lambda^2 w_1(\lambda, x) d\lambda$$

$$= \lim_{R\to\infty} \partial_t^2 \int_{-c-iR}^{-c+iR} e^{\lambda t} w_1(\lambda, x) d\lambda = (2\pi i)\partial_t^2 g_1(t, x).$$

Here in the last equality we take the derivative ∂_t^2 in the sens of distributions and we exchange ∂_t^2 and the limit $R \to \infty$. This implies

$$(\partial_t^2 - \Delta_x) g_1 = 0,\ t < 0, x \in \Omega.$$

Finally, taking into account (3.8) and (3.10) and applying the inverse Laplace transformation, for $t < 0$ we deduce

$$(\partial_\nu - \gamma\partial_t) g_1(t, x)|_\Gamma = \frac{1}{2\pi i}\int_{\mu=-c} e^{\lambda t} q(\lambda, x) d\lambda$$

$$= \frac{1}{2\pi i}\int_{\mu=-c} e^{\lambda t}\Big(\int_{-\infty}^0 e^{-\lambda s}(\partial_\nu - \gamma\partial_s) g_0(s, x)|_\Gamma ds\Big) d\lambda = (\partial_\nu - \gamma\partial_t) g_0(t, x)|_\Gamma.$$

Thus we obtain a solution $f(t, x) = g_0(t, x) - g_1(t, x)$ of the problem (1.3). Introduce the space $\mathcal{H}^m(\mathbb{R}_t^- \times \Omega)$ with norm

$$|||w(t, x)|||_m = \sum_{j+|\alpha|\leq m} \|\partial_t^j \partial_x^\alpha w(t, x)\|_{L^2(\mathbb{R}_t^- \times \Omega)}.$$

The estimate (3.11) implies $e^{c_{m-3/2}t} f(t, x) \in \mathcal{H}^{m-2}(\mathbb{R}_t^- \times \Omega)$ and we obtain the following

Proposition 3.2. *For $m > d/2+3$ there exists an operator*

$$P(t) : H^m_{comp}(\Omega) \longrightarrow e^{-c_{m-3/2}t}\mathcal{H}^{m-2}(\mathbb{R}_t^- \times \Omega) \tag{3.12}$$

satisfying

$$\begin{cases} (\partial_t^2 - \Delta_x)P(t) = 0,\ P(0) = 0,\ \partial_t P(0) = \mathrm{Id}, \\ (\partial_\nu - \gamma(x)\partial_t)P(t)|_{R_t^- \times \Gamma} = 0. \end{cases} \tag{3.13}$$

4. Absence of eigenvalues $|\lambda| \geq A_0$

To obtain an analytic continuation of the incoming resolvent for $|\lambda| \geq A_0$, we will follow the approach in section 4.6 in [2]. Let $\chi_a \in C_0^\infty(\mathbb{R}^d)$ be such that $\operatorname{supp} \chi_a \subset B(0,a)$ and $\chi_a \equiv 1$ for $|x| \leq \rho + 2\varepsilon$, $a \geq \rho + 3\varepsilon$, $\varepsilon > 0$. Introduce a function $\zeta_a(t,x) \in C^\infty$ such that

$$\zeta_a(t,x)|_{|x|>\rho+2\varepsilon} = \begin{cases} 1 \ \text{ for } 0 \geq t \geq -(|x| + T_a), \\ 0 \ \text{ for } t \leq -(x| + T_a + 1), \end{cases}$$

and $\zeta_a(t,x) = \varphi_a(t)$ for $|x| < \rho + \varepsilon$ with

$$\varphi_a(t) = \begin{cases} 1 \ \text{ for } 0 \geq t \geq -(\rho + \varepsilon + T_a), \\ 0 \ \text{ for } t \leq -(\rho + \varepsilon + T_a + 1). \end{cases}$$

Here $T_a \geq a$ is a fixed constant. Moreover, we suppose that $\zeta_a(t,x) = 1$ for $0 \geq t \geq -T_a$, $x \in \Omega$. Then for $|x| \leq \rho + \varepsilon$ we have $\partial_t \zeta_a(t,x) = \psi_a(t) \in C_0^\infty(\mathbb{R}_t^-)$.

For $m > d/2 + 3$ consider the operator $P(t)$ given by Proposition 3.2 and introduce

$$\zeta_a P(t)\chi_a : H^m(\Omega) \longrightarrow e^{-c_{m-3/2}t}\mathcal{H}^{m-2}(\mathbb{R}_t^- \times \Omega).$$

Therefore,

$$(\partial_t^2 - \Delta_x)(\zeta_a P(t)\chi_a) = [\Box, \zeta_a]P(t)\chi_a =: F_a(t)$$

and

$$(\partial_\nu - \gamma(x)\partial_t)(\zeta_a P(t)\chi_a)|_{\mathbb{R}_t^- \times \Gamma} = -\gamma(x)\psi_a(t)P(t)\chi_a\big|_{\mathbb{R}_t^- \times \Gamma} =: J_a(t).$$

Clearly, for $g \in H^m(\Omega)$ we have

$$\operatorname{supp}(F_a(t)g) \subset \{(t,x) \in \mathbb{R}_t^- \times \Omega :\ -(\max\{|x|, \rho+\varepsilon\} + T_a + 1) \leq t \leq -T_a)\},$$

$$\operatorname{supp}(J_a(t)g) \subset \{(t,x) \in \mathbb{R}_t^- \times \Gamma :\ -(\rho+\varepsilon+T_a+1) \leq t \leq -(\rho+\varepsilon+T_a)\}.$$

The operator $P(t)$ has a Schwartz kernel $p(t,x,y) \in \mathcal{D}'(\mathbb{R}_t^- \times \Omega \times \mathbb{R}_y^d)$ such that

$$(P(t)f)(x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)dy,\ f \in C_0^\infty(\mathbb{R}^d),$$

where the integral is taken in the sens of distributions. This implies that $J_a(t)$ has kernel $j_a(t,x,y) = \gamma(x)\psi_a(t)p(t,x,y)\chi_a(y)|_{x\in\Gamma}$. According to (3.13), we have

$$\gamma(x)\psi_a(t)P(t)\chi_a|_\Gamma : H^m(\Omega) \to \mathcal{H}^{m-5/2}(\mathbb{R}_t^- \times \Gamma).$$

We will construct a kernel $\tilde{b}_a(t,x,y)$ such that

$$\tilde{b}_a(t,x,y) = 0 \text{ for } -(\rho+\varepsilon+T_a+1) \leq t \leq 0,\ x \in \Gamma, \tag{4.1}$$

$$\partial_\nu \tilde{b}_a(t,x,y) = j_a(t,x,y) \text{ for } -(\rho+\varepsilon+T_a+1) \leq t \leq 0,\ x \in \Gamma. \tag{4.2}$$

To arrange the above conditions we apply Remark 2.1 considering the variables t, y as parameters. By the construction with local coordinates, we have $\tilde{b}_a(t,x,y) = \alpha(x)j_a(t,x,y)$ with some smooth function satisfying $\alpha(x)|_\Gamma = 0$, $\partial_\nu \alpha(x)|_\Gamma = 1$.

Choose $\chi_b \in C_0^\infty(B(0,a))$ such that $\chi_b = 1$ near $B(0,\rho+\varepsilon)$ and $\chi_a \equiv 1$ on the support of χ_b and define $b_a(t,x,y) = \chi_b(x)\tilde{b}_a(t,x,y)$. The conditions (4.1), (4.2) yield

$$(\partial_\nu - \gamma(x)\partial_t) b_a(t,x,y)|_{x\in\Gamma} = j_a(t,x,y).$$

Next let $B_a(t) : H^m(\Omega) \longrightarrow H^{m-2}(\mathbb{R}_t^- \times \Omega)$ be the operator with kernel $b_a(t,x,y)$. The smoothness of $b_a(t,x,y)$ follows form that of $j_a(t,x,y)$. The choice of the correction $B_a(t)$ leads to the equality

$$(\partial_\nu - \gamma\partial_t)\Big(\zeta_a P(t)\chi_a - B_a(t)\Big)\Big|_{x\in\Gamma} = 0$$

and we have

$$\text{supp}\,(B_a(t)g) \subset \{(t,x):\ -(\rho+\varepsilon+T_a+1) \le t \le -(\rho+\varepsilon+T_a),\ |x| \le a\}.$$

Denote by $W_a(t)$ the solution of the problem

$$\Box_0 W_a(t) = -(1-\chi_b)F_a(t), W_a(t) \equiv 0 \text{ for } t \ge 0. \tag{4.3}$$

Here $\Box_0 = \partial_t^2 - \Delta_0$ and Δ_0 is the Laplacian in $\mathbb{R}^d$. Repeating the argument of the proof of Theorem 4.3 in [2], it easy to see that there exists $C_a > T_a$ such that

$$\text{supp}\,(W_a(t)g) \subset \{(t,x) \in \mathbb{R}_t^- \times \Omega :\ -(|x|+C_a) \le t \le -(|x|+T_a)\}. \tag{4.4}$$

For completeness we present the details. Write

$$W_a(t) = -(1-\chi_b)\zeta_a H(t)P(t)\chi_a + H(t)U_0(t)((1-\chi_b)\chi_a) + Q_a(t),$$

where $U_0(t)$ is the solution of the Cauchy problem

$$\begin{cases}\Box_0 U_0 = 0, \\ U_0(0) = 0,\ (\partial_t U_0)(0) = (1-\chi_b)\zeta_a\chi_a.\end{cases}.$$

Then

$$\Box_0 Q_a = \Box_0 W_a + (1-\chi_b)\Box_0\Big(\zeta_a H(t)P(t)\chi_a\Big) + [\Delta,\chi_b]\zeta_a H(t)P(t)\chi_a$$

$$-\Box_0(H(t)U_0(t)(1-\chi_b)\chi_a)) = (1-\chi_b)\chi_a + [\Delta,\chi_b]\zeta_a H(t)P(t)\chi_a - (1-\chi_b)\chi_a$$

$$= [\Delta,\chi_b]\zeta_a H(t)P(t)\chi_a = r_{a,b}(t,x),$$

$$Q_a(0) = (\partial_t Q_a)(0) = 0.$$

We apply the Duhamel formula

$$Q_a(t) = -\int_t^0 U_0(t-s,x,y) r_{a,b}(s,y)dyds,$$

where $U_0(t,x,y)$ is the kernel of the operator $\frac{\sin(t\sqrt{-\Delta_0})}{\sqrt{-\Delta_0}}$. By the sharp Huyghens principle with a constant $C_a > 0$ we obtain

$$\text{supp}\,\Big(Q_a(t)g\Big) \subset \{(t,x):\ -(|x|+C_a) \le t \le 0\}.$$

Indeed, if $(t,x) \in \text{supp}\, Q_a(t)g$, there exist $(s,y) \in \text{supp}\, r_{a,b}$ and $t \le s \le 0$ such that $|x-y| = -(t-s)$ and this implies

$$-t \le |x| + |y| - s \le |x| + 2a + T_a + 1.$$

For $U_0(t)(1-\chi_b)\chi_a$ by a finite speed of propagation argument one obtains the same result. This proves the lower bound of t in (4.4). To establish the upper bound, we apply ones more the Duhamel formula for $W_a(t)$ writing

$$W_a(t)g = \int_t^0 U_0(t-s,x,y)(1-\chi_b(y))F_a(s)g)(y)dyds,\ t \leq 0.$$

If $(t,x) \in \text{supp}\, W_a(t)g$, there exist $(s,y) \in \text{supp}\,(1-\chi_b)F_a(s)g(y)$ and $t \leq s \leq 0$ such that $|x-y| = -(t-s)$. Hence

$$-t \geq |x| - |y| - s \geq |x| - |y| + |y| + T_a = |x| + T_a.$$

Choose a function $\chi_c \in C_0^\infty(B(0,a))$ which is equal to 1 near $B(0,\rho)$ and such that $\chi_b \equiv 1$ on supp χ_c and introduce the approximative resolvent

$$R_a^\sharp(\lambda) = \int_{-\infty}^0 e^{-\lambda t}\Big[\zeta_a P(t)\chi_a + (1-\chi_c)W_a(t) - B_a(t)\Big]dt,\ \text{Re}\,\lambda < 0.$$

The integral is well defined since for fixed x the supports with respect to t of the functions under integration are bounded. For $m > d/2+3$ we have $R_a^\sharp(\lambda)$: $H^m_{comp}(\Omega) \to H^{m-2}_{loc}(\Omega)$. We obtain

$$(\Delta - \lambda^2)R_a^\sharp(\lambda) = \chi_a\Big(\text{Id} + L_a(\lambda)\Big),\ \ (\partial_\nu - \lambda\gamma(x))R_a^\sharp(\lambda)|_{x\in\Gamma} = 0$$

with

$$L_a(\lambda) = \int_{-\infty}^0 e^{-\lambda t}\Big(-\chi_b F_a(t) - [\Delta, \chi_c]W_a(t) + \Box B_a(t)\Big)dt.$$

After an integration by parts with respect to t we deduce

$$L_a(\lambda) = \lambda^{-1}\int_{-\infty}^0 e^{-\lambda t}\Big(-\chi_b\partial_t F_a(t) - [\Delta, \chi_c]\partial_t W_a(t) + \partial_t\Box B_a(t)\Big)dt.$$

For the invertibility of $\text{Id} + L_a(\lambda)$ in $H^m(\Omega\cap B(0,a))$ we must estimate the $\|.\|_{m,a}$ norms of the terms under integration. For $\partial_t F_a(t)$ one observes that $\partial_t^2 P(t) = \Delta P(t)$, hence $\|\partial_t F_a(t)\|_{m,a} \leq C_{1,m}\|\chi_a\|_{m+4}, a$. Similarly, for $\Box B_a(t)$ we exploit once more the property of $P(t)$ since $b_a(t,x,y) = \chi_b(x)\alpha(x)p(t,x,y)$ and conclude that $\|\partial_t\Box B_a(t)\|_{m,a} \leq C_{2,m}\|\chi_a\|_{m+4,a}$, Finally, $\partial_t W_a(t)$ can be estimated by $F_a(t)$ and we deduce

$$\|L_a(\lambda)\|_{m,a} \leq D_{m+4,a}|\lambda|^{-1}\|\chi_a\|_{m+4,a}.$$

Let $\rho < a_1 < a$. We fix $m > d/2+3$ and fix a function $\chi_a \in C_0^\infty(B(0,a))$ such that $\chi_a \equiv 1$ for $|x| \leq a_1$. Choose $|\lambda| \geq 2D_{m+4,a}\|\chi_a\|_{m+4,a} = A_0$. Then the operator $(\text{Id} + L_a(\lambda))^{-1}$ is invertible in $H^m(\Omega\cap B(0,a))$ and for every $g \in H^m(\Omega)$ with supp $g \subset B(0,a_1)$ by an analytic continuation we get

$$R_a(\lambda)g = R_a^\sharp(\lambda)(\text{Id} + L_a(\lambda))^{-1}g = \chi_a g = g,\ \text{Re}\,\lambda < 0,\ |\lambda| \geq A_0.$$

The function $u(x,\lambda) = R_a(\lambda)g$ is analytic solution of (2.4) with $f = 0$. According to Lemma 2.1, this implies that for every $s \geq 0$ the operator $\mathcal{C}(\lambda)^{-1} : H^s(\Gamma) \to H^{s+1}(\Gamma)$ is analytic for $\text{Re}\,\lambda < 0$, $|\lambda| \geq A_0$ and we obtain the following

Proposition 4.1. *There exists $A_0 > 0$ such that the operator G has no eigenvalues*

$$\lambda \in \{\mathbb{C} :\ \text{Re}\,\lambda \leq -c_0\} \cup \{\mathbb{C} :\ \lambda \in \mathbb{C}_-,\ |\lambda| \geq A_0\}.$$

It is important to note that $D_{m+4,a}$ depends of constants $A_{m+4}, B_{m+4}, K_{m+4}$ in subsection 3.2.

5. Absence of eigenvalues $|\lambda| \leq A_0$

Consider for $\varepsilon \in [0, 1]$ the operator

$$\mathcal{C}_\varepsilon(\lambda) := N(\lambda) - \lambda\varepsilon\gamma(x) = \Big(\mathrm{Id} - \lambda\varepsilon\gamma(x)N(\lambda)^{-1}\Big)N(\lambda).$$

Let e^{tG_ε} be the semi-group with generator G_ε defined with boundary condition $(\partial_\nu - \varepsilon\gamma(x)\partial_t)f|_\Gamma = 0$. Obviously, $\max_{x\in\Gamma}\varepsilon\gamma(x) = \varepsilon m_0 < 1$. Set $\alpha_\varepsilon = 1 - \varepsilon m_0 \geq \alpha$. Below for simplicity we write H^s instead of $H^s(\Gamma)$. It is easy to see that for $\mu \leq -c_0$ we have

$$-\operatorname{Re}((N(\lambda) - \lambda\varepsilon\gamma(x))\varphi, \varphi)_0 \geq (\alpha_\varepsilon|\mu| - C_0)\|\varphi\|_0^2, \ \forall\varphi \in C^\infty(\Gamma) \tag{5.1}$$

with constant $C_0 > 0$ independent of ε and μ. By the argument in subsection 3.1 we conclude that the operator $\mathcal{C}_\varepsilon(\lambda)^{-1} : H^s \to H^{s+1}$ is analytic for $\mu \leq -c_0$ uniformly with respect to ε. Recall the constants $c_m = \max\left\{\frac{2C_m}{\alpha}, c_{m-1}\right\} > 0$ defined in subsection 3.1. Since C_m in (3.2) are independent of $\gamma(x)$, we obtain

$$c_{m,\varepsilon} = \max\left\{\frac{2C_m}{\alpha_\varepsilon}, c_{m-1,\varepsilon}\right\} \leq c_m.$$

Moreover, for $\mu \leq -c_m$ we have estimates

$$\|\mathcal{C}_\varepsilon(\lambda)^{-1}h\|_s \leq \frac{B_s}{\alpha_\varepsilon|\mu|}\|h\|_s \leq \frac{B_s}{\alpha|\mu|}, \ \forall h \in H^s(\Gamma) \tag{5.2}$$

with $B_s > 0$ independent of ε. To do this, we repeat the argument of subsection 3.1 and recall that the constants A_m depend of the norms of the derivatives of $\varepsilon\gamma(x)$ of order $j \leq m$, hence they can be chosen to be independent of ε. Repeating the proof of subsection 3.2, we construct an operator

$$P_\varepsilon(t) : H^m_{comp}(\Omega) \longrightarrow e^{-c_{m-3/2}t}\mathcal{H}^{m-2}(\mathbb{R}_t^- \times \Omega)$$

satisfying

$$\begin{cases}(\partial_t^2 - \Delta_x)P_\varepsilon(t) = 0, \ P_\varepsilon(0) = 0, \ \partial_t P_\varepsilon(0) = \mathrm{Id}, \\ (\partial_\nu - \varepsilon\gamma(x)\partial_t)P_\varepsilon(t)|_{R_t^- \times \Gamma} = 0.\end{cases} \tag{5.3}$$

The analysis of Section 4 implies that $\mathcal{C}_\varepsilon(\lambda)^{-1}$ has an analytic continuation for $\operatorname{Re}\lambda < 0$, $|\lambda| \geq A_0$, where $A_0 > 0$ is independent of ε. This follow from the fact that the constants $A_{m+4}, B_{m+4}, K_{m+4}$ in subsection 3.2 can be chosen uniformly with respect to ε. Denote by $A(\lambda, \varepsilon)$ the compact operator

$$A(\lambda, \varepsilon) = \lambda\varepsilon\gamma(x)N(\lambda)^{-1} : \ H^{1/2} \to H^{3/2}.$$

Consider the domain $\omega \subset \bar{C}_-$ bounded by the the contour

$$\zeta = \{z \in \mathbb{C} : \operatorname{Im} z = -A_0, \ -c_0 \leq \operatorname{Re} z \leq 0\} \cup \{z \in \mathbb{C} : \operatorname{Re} z = 0, \ |\operatorname{Im} z| \leq A_0\}$$

$$\cup \{z \in \mathbb{C} : \operatorname{Im} z = A_0, \ -c_0 \leq \operatorname{Re} z \leq 0\} \cup \{z \in \mathbb{C} : \operatorname{Re} z = -c_0, \ |\operatorname{Im} z| \leq A_0\}.$$

For sufficiently small ε the operator $C_\varepsilon(\lambda)$ is invertible for $\lambda \in \omega$. Indeed, for all $\lambda \in \omega$ we have $|\lambda| \leq B_0$ and $\|\gamma(x)N(\lambda)^{-1}\|_{H^{1/2}\to H^{1/2}} \leq B_1$. Then for $\varepsilon \leq \frac{1}{2B_0B_1} = \eta_0$ one arranges

$$\|\lambda\varepsilon\gamma(x)N(\lambda)^{-1}\|_{H^{1/2}\to H^{1/2}} \leq 1/2, \ \lambda \in \omega.$$

This implies that $(\mathrm{Id} - A(\lambda, \varepsilon))$ is invertible in $H^{1/2}$ and $\mathcal{C}_\varepsilon(\lambda)^{-1}$ is analytic for $\lambda \in \omega$. Define

$$\eta = \sup\{\varepsilon : \ \varepsilon \in [0, 1] : \ \mathcal{C}_\varepsilon(\lambda)^{-1} \text{ is analytic for } \lambda \in \omega\}.$$

Obviously, $\eta_0 \leq \eta \leq 1$. Notice that if $\mathcal{C}_\varepsilon(\lambda)$ and $\mathcal{C}_\eta(\lambda)$ are invertible, we have the equality

$$\mathcal{C}_\varepsilon(\lambda)^{-1}\Big(\mathrm{Id} - (\varepsilon - \eta)\lambda\gamma\mathcal{C}_\eta(\lambda)^{-1}\Big) = \mathcal{C}_\eta(\lambda)^{-1}. \tag{5.4}$$

We have two cases: (i) $\eta = 1$, (ii) $\eta < 1$. If $\eta = 1$ and $\mathcal{C}(\lambda) = \mathcal{C}_1(\lambda)$ is invertible for $\lambda \in \omega$, the operator G has no eigenvalues in ω. In the case (ii) if $\mathcal{C}_\eta(\lambda)$ is invertible for $\lambda \in \omega$, by using (5.4), we obtain that $\mathcal{C}_\varepsilon(\lambda)$ is invertible for $1 > \varepsilon > \eta$, $\lambda \in \omega$ and $\varepsilon - \eta$ sufficiently close to 0. To prove this, for $\varepsilon - \eta > 0$ sufficiently small and $\mathcal{C}_\varepsilon(\lambda)^{-1}$ analytic, we write

$$\mathcal{C}_\varepsilon(\lambda)^{-1} = \mathcal{C}_\eta(\lambda)^{-1}\Big(\mathrm{Id} - (\varepsilon - \eta)\lambda\gamma\mathcal{C}_\eta(\lambda)^{-1}\Big)^{-1}. \tag{5.5}$$

The right hand part remains bounded for $\lambda \in \omega$ and $\varepsilon - \eta > 0$ small and we deduce the analyticity of $\mathcal{C}_\varepsilon(\lambda)^{-1}$ for $\lambda \in \omega$ by analytic continuation. This implies a contradiction with the definition of η. Consequently, in both cases (i) and (ii) we can assume that $\mathcal{C}_\eta(\lambda)^{-1}$ has a pole $\lambda_0 \in \mathring{\omega}$, where $\mathring{\omega}$ denotes the interior of ω.

Choose a small $\delta > 0$ so that $\{z \in \mathbb{C} : |z - \lambda_0| \leq \delta\} \subset \mathring{\omega}$ and $\mathcal{C}_\eta(\lambda)^{-1}$ has no poles λ with $0 < |\lambda - \lambda_0| \leq \delta$. Let $\gamma_0 = \{z \in \mathbb{C} : |z - \lambda_0| = \delta\}$ be positively oriented. Consider the projector

$$\Pi_\eta(\lambda_0) = \frac{1}{2\pi\mathbf{i}}\int_{\gamma_0}(\lambda - G_\eta)^{-1}d\lambda$$

having rank equal to the multiplicity of the eigenvalue λ_0 of G_η. With a constant $D_0 > 0$ independent of λ one has a bound $\|(G_\eta - \lambda)^{-1}\|_{\mathcal{H}} \leq D_0$, $\lambda \in \gamma_0$. We claim that for $0 < \varepsilon < \eta$ and $\eta - \varepsilon$ sufficiently small we have

$$\|(G_\eta - \lambda)^{-1} - (G_\varepsilon - \lambda)^{-1}\|_{\mathcal{H}} \leq C_0|\varepsilon - \eta|,\ \forall \lambda \in \gamma_0. \tag{5.6}$$

with $C_0 > 0$ independent of ε and λ. This follows from a perturbation argument (see Section 5 in [1]). We present a direct proof. First, for $\lambda \in \gamma_0$ the operator $\mathcal{C}_\eta(\lambda)$ is invertible and $\|\gamma(x)\mathcal{C}_\eta(\lambda)^{-1}\|_{H^{1/2}\to H^{3/2}} \leq D_1$, $\lambda \in \gamma_0$. Applying (5.5) for $|\varepsilon - \eta| \leq \frac{1}{2B_0D_1}$ and $\lambda \in \gamma_0$ we deduce a bound

$$\|\mathcal{C}_\varepsilon(\lambda)^{-1}\|_{H^{1/2}\to H^{1/3}}$$
$$\leq \|\mathcal{C}_\eta(\lambda)^{-1}\|_{H^{1/2}\to H^{3/2}}\Big\|\Big(\mathrm{Id} - (\varepsilon - \eta)\lambda\gamma\mathcal{C}_\eta(\lambda)^{-1}\Big)^{-1}\Big\|_{H^{1/2}\to H^{1/2}} \leq 2D_1.$$

Taking into account the bounds for $\mathcal{C}_\eta(\lambda)^{-1}$ and $\mathcal{C}_\varepsilon(\lambda)^{-1}$, from (5.4) we get

$$\|\mathcal{C}_\eta(\lambda)^{-1} - \mathcal{C}_\varepsilon(\lambda)^{-1}\|_{H^{1/2}\to H^{3/2}} \leq 2|\varepsilon - \eta|B_0D_1^2,\ \lambda \in \gamma_0.$$

Next we exploit the representation (2.3) for the first components $u_\eta(x, \lambda)$ and $u_\varepsilon(x, \lambda)$ of $(G_\eta - \lambda)^{-1}(f, g)$ and $(G_\varepsilon - \lambda)^{-1}(f, g)$ with $\mathcal{C}_\eta(\lambda)^{-1}$ and $\mathcal{C}_\varepsilon(\lambda)^{-1}$, respectively. Choose a function $\varphi \in C_0^\infty(\mathbb{R}^d)$ equal to 1 for $|x| \leq a$, $a > \rho$. For $\lambda \in \gamma_0$ we obtain

$$\Big\|(\mathcal{C}_\eta(\lambda)^{-1} - \mathcal{C}_\varepsilon(\lambda)^{-1})\Big(\partial_\nu R_D(\lambda)(g + \lambda f)|_\Gamma + \gamma f|_\Gamma\Big)\Big\|_{H^{3/2}}$$
$$\leq |\varepsilon - \eta|B_0D_2D_1^2\Big(\|\varphi R_D(\lambda)(g + \lambda f)\|_{H^1(\Omega)} + \|\varphi f\|_{H^1(\Omega)}\Big).$$

We prove the claim (5.6) by using the estimates for the resolvent $R_D(\lambda)$ and the operator $K(\lambda)$. Next for $\eta - \varepsilon > 0$ sufficiently small we obtain

$$\Big\|\frac{1}{2\pi\mathbf{i}}\int_{\gamma_0}\Big((\lambda - G_\eta)^{-1} - (\lambda - G_\varepsilon)^{-1}\Big)d\lambda\Big\|_{\mathcal{H}} \leq C_0\delta|\varepsilon - \eta|. \tag{5.7}$$

The analyticity of $\mathcal{C}_\varepsilon(\lambda)^{-1}$ for $\varepsilon < \eta$ implies $\int_{\gamma_0}(\lambda - G_\varepsilon)^{-1}d\lambda = 0$. For small $\eta - \varepsilon$ the estimate (5.7) leads to $\|\Pi_\eta(\lambda_0)\|_{\mathcal{H}\to\mathcal{H}} < 1$ and this yields a contradiction with the existence of the pole λ_0. This completes the proof of Theorem 1.1.

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